BANK OF ENGLAND

# Staff Working Paper No. 894

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November 2020

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## Abstract

This paper proposes a positive theory of the link between banks' capitalisation and their liquidity-risk taking as well as the severity of fire-sale problems and liquidity crises. In the basic framework of an individual bank's decisions, we find that banks' incentives to hold liquidity for precautionary reason are increasing with their capital. In a continuum-of-banks setting in which both precautionary and speculative motives of liquidity holdings are taken into account, we find that while the fire-sale discount is decreasing with the capitalisation of the banking system, the link between the latter and the severity of liquidity crises is not monotonic.



**Key words:** Leverage, Precautionary liquidity holdings, speculative liquidity holdings, wholesale debts, cash-In-the-market pricing.

**JEL classification:** D82, G21.


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A previous version was circulated under the title: 'Liquidity Management in Banking: What is the Role of Leverage?'. We are grateful to Toni Ahnert (discussant), Urs Birchler, Jean-Edouard Colliard, Gianni De Nicolo, Hans Gersbach, Michel Habib, Antoine Lallour, Frederic Malherbe, Manuel Munoz (discussant), Ettore Panetti (discussant), Bruno Parigi, Sebastian Pfeil, Jean-Charles Rochet and Tanju Yorulmazer (discussant) for their helpful comments and suggestions. We also thank seminar and conference participants at 2019 BCBS RTFCEPR join Workshop in Basel, Switzerland; 27th Finance Forum in Madrid, Spain; IWFSAS 2018 in London, UK; ESEM 2018 in Cologne, Germany; 2018 IBEFA Summer conference in Vancouver, Canada; the first annual workshop of the ESCB Research Cluster 3 on financial stability, macroprudential regulation and microprudential supervision in Athens, Greece; FMA 2017 in Boston, US; EEA 2017 in Lisbon, Portugal; EFMA 2017 in Athens, Greece; RES 2017 in Bristol, UK; IFABS 2016 in Barcelona, Spain; 2015 Paris Financial Management Conference in Paris, France; the University of Zurich and the University of Basel, for their useful feedback. A significant part of this research was done when the second author was affiliated with the University of Zurich and received funding from the ERC (the grant agreement 249415-RMAC), from NCCR FinRisk (project 'Banking and Regulation') and from Swiss Finance Institute (project 'Systemic Risk and Dynamic Contract Theory'). The views expressed in this paper are those of the authors, and not necessarily those of the Bank of England or its committees.

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ISSN 1749-9135 (on-line)

# 1 Introduction

In the aftermath of the global financial crisis, two new liquidity standards, namely the Liquidity Coverage Ratio (LCR) and the Net Stable Funding Ratio (NSFR), were introduced by the Basel Committee. Their main objectives encompass creating incentives for banks to better manage their liquidity risk and improving the banking sector's ability to absorb liquidity shocks. These new policy measures raise a number of questions, including the potential substitutability and complementarity between them and the already-existing capital requirements. Using a positive approach, the current paper aims to shed light on this question by examining the effects of banks' leverage on their incentives with respect to liquidity risk management. Precisely, this paper explores the following questions: Do better capitalised banks have better incentives to manage their liquidity risk? Are better capitalised banking systems more or less vulnerable to liquidity crises? How does the leverage distribution of the banking system affect the extent of the fire-sale problem? It thus proposes a positive theory of the link between banks' capitalisation and liquidity risk taking, as well as the severity of the fire-sale problem and liquidity crises.

We develop this theory in a model where banks engage in maturity transformation, borrowing short and lending long. This leaves them vulnerable to liquidity shocks. So far, in the literature on bank liquidity, the liquidity shocks are usually modelled a la Diamond - Dybvig[1]. In this way, banks are assumed to be financed by retail demandable deposits and thus, may be hit by a liquidity shock if a high number of their depositors come to withdraw. This withdrawal is, in turn, modelled as being determined either by the time preferences of risk-averse depositors or by the coordination problem between them each receiving some *private* signal about the quality of the banks' assets. In this paper, inspired by several observations from the 2007 - 2009 crisis, we adopt a modelling approach that differs in two main aspects.

First, in our setup, banks' short-term debts (instead of being retail deposits) take the form of unsecured wholesale debts, such as unsecured commercial papers. These types of debt were at the centre of the recent global financial crisis. Since unsecured wholesale

[1]See for example Allen and Gale (1998)

debts are usually held by sophisticated investors, such as financial institutions or money market funds, we assume that the banks' short-term debtholders are risk-neutral, and that the debt repayment is endogenously determined, depending on the banks' choice of assets. This endogeneity is a key difference between our paper and other papers that model retail deposits and assume an exogenous deposit rate.

Second, the liquidity event is modelled by the arrival of unfavorable and *public* information on the quality of banks' assets. This new information negatively affects banks' funding liquidity, and thus, makes it difficult for banks to meet their repayment obligations. This is analogous to the unfolding events of the recent crisis, according to which the triggering of the liquidity problem involved some public information regarding an increase in subprime mortgage defaults. What was followed was a deterioration of the short-term funding market, such as the commercial paper market.

Hence, the context we have in mind is one of banks that are financed by equity and wholesale unsecured short-term debt that matures after one period. Banks could invest in two types of assets. One is short-term assets, referred to as liquid assets, and the other is long-term assets. The latter is more profitable than the former, but takes two periods to yield a cash flow. This maturity mismatch between the payoff of the long-term assets and debt repayments gives rise to a need for banks to arrange for some liquidity at the interim date when their short-term debt repayments are due. We assume that banks could raise liquidity at the interim date by pledging future cash flows of their long-term assets - *funding liquidity*. However, the banks' capacity to generate liquidity in this way may be restricted if bad news on the quality of the long-term assets is revealed at the repayment date. This limited pledgeability provides, in our model, a reason for banks to invest in the less profitable, but more liquid short-term assets if they want to insure themselves against liquidity shocks.

We start with an examination of banks' incentives to hold liquid assets to protect themselves against future liquidity shocks, i.e., incentives for precautionary liquidity holdings, and how these incentives are affected by banks' leverage. This is done within a simple framework of an individual bank's decisions. Then, in order to analyse the link between the capitalisation of the banking system and the extent of the fire-sale problem and liquidity crises, we cast this building block of individual banks' decisions into a continuum-of-banks setting in which we account for the existence, at the interim date, of a secondary market for long-term assets. Therefore, banks with a liquidity shortage could sell their long-term assets in order to raise liquidity. This additional element serves two purposes. It first allows us to capture another source of liquidity that banks can rely on, namely *market liquidity*. It also enables us to take into consideration another motive driving banks' choice of ex-ante liquidity holdings, in addition to a precautionary motive. That is the "strategic" motive of being able to take advantage of fire sales - the so-called *speculative motive of liquidity holdings*[2].

In our continuum-of-banks setting, asset sales are modelled as in Acharya and Viswanathan (2011). We assume that the long-term assets are specific and can only be acquired by banks that survive liquidity shocks and have spare liquidity. Therefore, the price of long-term assets, which is determined by the market-clearing condition, is of the "cash-in-the-market" type proposed by Allen and Gale (1994). Moreover, we assume that the returns of long-term assets are perfectly correlated across banks. Hence, the new information will touch on the assets of all banks simultaneously, which means that the liquidity shock in our setup takes the form of systemic shock.

Our contribution is twofold. First, we highlight a new channel that links banks' capitalisation and their incentives for precautionary liquidity holdings. Our main finding is that a well capitalised bank will have incentives to hold an adequate level of liquid assets to shield itself from liquidity shocks. The intuition lies in the fact that when leverage is high, the banks' exposure to liquidity shocks is large. Buying insurance by securing some ex-ante liquidity holdings is then too costly, which induces banks to forgo the insurance option and gamble. Two interesting implications result from this finding.

From the perspective of policy implications, in our simple setting, there exists a cutoff capital ratio level above which banks will choose to manage their liquidity risk prudently. This implies that a properly designed capital requirement could provide banks with the right incentives to have adequate liquidity holdings. Hence, in our framework, capital and liquidity requirements are, at least to some extent, substitute in the sense that highly capitalised banks will automatically choose their asset mix to meet liquidity and funding

[2]See Acharya et al. (2011)

requirements.[3] We are not claiming that this result implies that, in the presence of capital requirements, liquidity requirements such as LCR are redundant.[4] All we are claiming is that a restriction on banks' leverage can have a positive impact on their incentives to manage their liquidity risk.

From the conceptual perspective, our finding highlights the difference in the condition that is necessary for the existence of the linkage between banks' capital and their liquidity risk taking, as compared to the condition for the linkage between banks' capital and their credit-risk taking. Note that in our model, the debt repayment is endogenously determined to make banks' debtholders break-even in expected terms. Therefore, differently from the case of credit risk, the effect of banks' capital on their liquidity risk taking does not arise due to the failure of banks' creditors to properly price the level of liquidity risk taken by banks into the required debt repayments.

In relation to our second contribution, to the best of our knowledge, this paper is the first one that takes into account, in a unified setting, all three possible sources of liquidity, as well as two motives for banks to hold liquid assets and examines the implications of their interaction for a fire-sale discount and the severity of liquidity crises. We find that when the banking system becomes more highly leveraged, a *precautionary motive of liquidity holdings* leads to a weak decrease in the equilibrium price of long-term assets and an increase in the fraction of banks that fail subsequent to the materialisation of a liquidity shock - our measure of the severity of liquidity crises. However, a *speculative motive of liquidity holdings* has the opposite effect. The overall impact of a change in the capitalisation of the banking system differs between the fire-sale discount and the fraction of failed banks. We show analytically that the fire-sale discount is weakly increasing with the degree of leverage in the banking system. However, and interestingly, our numerical analysis highlights that the proportion of banks that will fail when a liquidity shock is materialised is not monotonic with respect to the capitalisation of the banking system.

[3] In this sense, in our setup, there is no cost in having a liquidity requirement in addition to a capital requirement. This is because in this paper, we abstract from the positive impact of banks' capital on the incentives of creditors to run and on banks' ability to refinance their short-term debts. This channel is also potentially important channel for the substitutability between capital and liquidity requirements, which would call for dialling down liquidity and funding requirements if dialling up capital requirements would allow while keeping the level of resilience intact.

[4] In our view, the redundancy question needs to be addressed in a general equilibrium setting.

Improving the banking system's capitalisation is beneficial, except when the system is poorly capitalised. Moreover, the difference in the impact of the leverage distribution of the banking sector on the fire-sale discount and on the fraction of failed banks suggests that a severe fire-sale problem and a high proportion of bank failures in the system do not necessarily happen simultaneously.

The organization of the paper is as follows. After discussing the related literature in the next section, we analyse in Section 3 the banks' choice of precautionary liquidity holdings and the effect of banks' leverage on this decision. Then we move on, examining in Section 4 the link between the leverage distribution of the banking system and the fire-sale problem, as well as the severity of the liquidity crises. Finally we conclude in Section 5. All proofs are provided in the Appendix.

# 2 Related Literature

This paper contributes to the literature on bank liquidity by considering the effect of banks' leverage on their incentives to hold liquidity. As such, it is directly related to several papers that study banks' choice of investment between liquid and illiquid assets. They differ in the determinants upon which they focus. Acharya et al. (2011a) examine the effect of policy interventions to resolve bank failure on ex-ante bank liquidity holdings. Malherbe (2014) provides a model in which the fear of future market illiquidty due to adverse selection may trigger hoarding behavior today. Heider et al. (2015) analyse banks' liquidity holdings to shed light on how banks' private information about the risk of their assets affects the trading and pricing of liquidity in the interbank market. Acharya et al. (2015) study how the liquidity choices of firms are shaped by risk-sharing opportunities in the economy. Our paper considers the effect of banks' liability structure on their choices of liquidity holdings.

The current paper is also connected to several contributions that use the "cash-in-the-market-pricing" mechanism proposed by Allen and Gale (1994, 2004, 2005) to understand the financial fragility (see e.g., Bolton et al. (2011), Acharya and Viswanathan (2011b), Freixas et al. (2011) and Gale and Yorulmazer (2013)). The most closely related paper to our work is that of Acharya and Viswanathan (2011b), which builds a model to understand the de-leveraging of the financial sector during crises. They examine how adverse shocks that materialize during good economic times, represented by high expectations about economic fundamentals, lead to greater de-leveraging and asset price deterioration. Our continuum-of-banks setup with asset sales is, in fact, inspired by Acharya and Viswanathan (2011b)'s setting. The main difference is that we allow banks to hold liquidity ex-ante to self-insure against liquidity shocks, which enables us to shed light on how banks' incentives to manage liquidity risk are affected by their liability structure.

Furthermore, the insights on which our model is built are linked to several other literatures. Indeed, the idea that the liability structure of a bank may impact its choices of asset composition is linked to the extensive literature that evaluates the foundation for the imposition of capital regulation. See, among others, Rochet (1992), Besanko and Kanatas (1996), Blum (1999) and Repullo (2004).[5] This literature examines how banks' incentives to take excessive risk can be curbed by requiring banks to maintain an adequate capital ratio. While the focus of this literature is the impact on the banks' incentives to take credit risk, our paper aims to examine the effect on their incentives to manage their liquidity risk.

In our paper, the reason for banks to hold liquidity is based on two assumptions: ex-ante uncertainty about liquidity needs and limited pledgeability due to asymmetric information. These two assumptions are similar to those used by Hölmstrom and Tirole (1998) to analyse the liquidity demand of the corporate sector and the role of government in supplying liquidity. The main difference lies in the fact that in Hölmstrom and Tirole (1998), liquidity shocks arise as production shocks to the firms' technologies. The size of the shocks is exogenous and especially independent of the firms' balance sheet characteristic. We rather derive liquidity needs as being determined in equilibrium by an asset-liability mismatch. Such a difference explains why in Hölmstrom and Tirole (1998), the firms' liquidity demand does not depend on their liability structure, whereas in our framework, it does. We believe that liquidity shocks arising from technology shocks, as in Hölmstrom and Tirole (1998), are suitable for non-financial enterprise, while our formulation is more reasonable in the context of financial institutions.

Finally, a number of recent papers have focused on the optimal design of bank liquidity

[5]For an excellent review of this literature, see Freixas and Rochet (2008) and VanHoose (2007).

regulation (see e.g., Calomiris et al. (2014), Walther (2016), Santos and Suarez (2016) and Kashyap et al. (2017)). All of these papers present different rationales for introducing liquidity requirements. Among these papers, the closest one to ours is Kashyap et al. (2017), as they consider the interaction of bank's liquidity and leverage decisions. They provide a rationale for capital and liquidity requirements by showing that there is a wedge between the optimal liquidity and leverage choices made by a bank and a social planner. In their setting, there is no room for speculative liquidity holdings, as there is not an interbank market for the long term asset. In our paper, we take the leverage structure as given, but analyze how it shapes banks' liquidity holdings considering both the precautionary and speculative motive for liquidity holdings. This allows as to draw conclusions on how banks' leverage has an impact on fire sales prices.

# 3 Precautionary liquidity holdings and leverage

In this section, we study the impact of banks' leverage on their precautionary liquidity holdings in a simple setting of an individual bank that seeks to manage its liquidity risk.

## 3.1 Setup

There are three dates $t = 0, 1, 2$ and a bank with balance sheet of size normalised to 1. We assume that at date 0, the bank has a proportion $E$ of funds as equity and raises the remaining fraction $1 - E$ by issuing unsecured short-term debt to risk-neutral investors.[6] The face value of the short-term debt that needs to be repaid at date 1 is denoted by $D_1$ and will be endogenously determined.

**Investment opportunities.** The bank has access to two investment opportunities. The first one is a short-term asset, referred to as a liquid asset, which produces a gross deterministic return of 1 per period. The second investment opportunity is a constant-

[6]The academic literature has offered two explanations about why banks use short-term debt. The first one comes from the beneficial effects of short-term debt in disciplining banks' managers. The second explanation focuses on the role of banks as liquidity providers: banks issue short-term debt to provide flexibility to creditors who may be hit by a liquidity shock. In the current paper, we do not explicitly model the reason for which the bank uses short-term funding, but assume it exogenously. In line with the second explanation, we justify such use as a bank's response to the investors' demand for liquid investment.

return-to-scale project, referred to as a long-term asset, with two main features. First, it generates a random cash flow $\tilde{y} \geq 0$ only at date $t = 2$. Second, its returns are exposed to a shock that is realised at date $t = 1$, as described below.

**Liquidity shock.** At date 1, new information regarding the returns of the long-term asset becomes publicly available. Bad news is revealed with the probability $1 - \alpha$ and good news happens with the complementary probability. Note that although the new information is publicly observable, it is not verifiable, which implies that the short-term debt repayment cannot be contingent upon it. We will, hereafter, refer to the revelation of bad news as the materialisation of a liquidity shock since it will limit the extent to which the bank can pledge the future cash flow of the long-term asset.

Indeed, we assume that in the case of positive information, the long-term asset yields at date 2 a payoff equal to $y_H > 0$ per unit of investment when it succeeds - which occurs with the probability $\theta$ - and zero when it fails. Negative information has two implications for a long-term asset's return. First, the unit cash flow $y_L$ generated by this asset in the case of success is lower (i.e., $y_L < y_H$). Moreover, the success probability in this situation depends on bank's monitoring, denoted by $m$, which is not observable by outsiders. For simplicity, we assume that the bank can choose either to exert monitoring effort, corresponding to $m = 1$, or to shirk, corresponding to $m = 0$. If it does exert effort, the probability of success is equal to $\theta$, as in the case of positive information. However, if it shirks, the probability of success is reduced to $\theta - \Delta$. Monitoring is costly for the bank, and we capture it by assuming that the bank obtains some private benefit $B$ per unit of long-term asset if it shirks. Figure 1 summarizes the payoff structure of the long-term asset.

**Timing.** The sequence of events, which is summarised in Figure 2, is as follows. At $t = 0$, the bank decides how much to invest in each of the assets. Denote by $c$ the proportion of liquid assets held by the bank; the remaining proportion $1 - c$ is invested in the long-term asset.[7] At $t = 1$, the information regarding the quality of the long-term

[7] In the current setup, we assume that the total size of the bank's balance sheet is fixed and normalised to 1. Hence, if the bank chooses to invest a fraction $c$ of its balance sheet in liquid assets, the remaining fraction $1 - c$ will be in the form of long-term assets. Alternatively, one could assume a fixed size $I$ of investment in long-term assets and the bank can choose the volume of liquid assets, expressed as a multiple $n$ of $I$, held on top of that. In this case, the fraction of liquid assets over total assets is $n/1 + n$. All our results will carry over with $c$ replaced by $n/1 + n$.

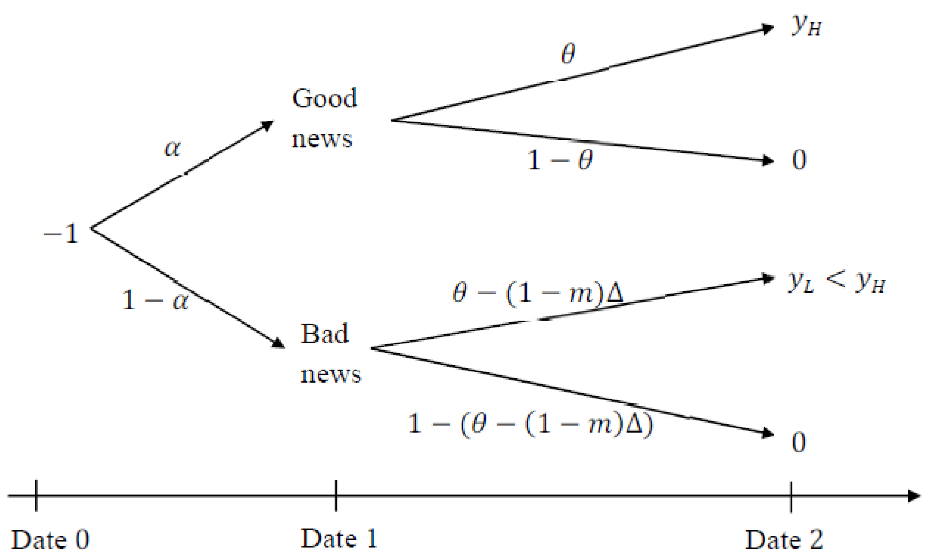



Figure 1: Payoff of the long-term asset

asset is revealed, and the short-term debt contracts mature. If the bank's holdings of liquid assets are not enough to repay debtholders, the bank will issue new debt pledging the future payoff of the long-term asset. In case the bank cannot borrow enough to repay debtholders, the bank is liquidated. At date $\frac{3}{2}$, between $t = 1$ and $t = 2$, if necessary, the bank decides whether to exert effort to monitor the long-term asset. At $t = 2$, the long-term asset returns are realised, and all payments are settled.
We assume that in case the bank is being liquidated, the liquidation value per unit of long-term asset is equal to $\ell$, regardless of the information revealed at date 1.

Note that in our framework, it does not matter whether the holders of the new short-term debt issued at date 1 is the current bank's debtholders or new investors. What matters is that the price of this new short-term debt depends on the information revealed. If new debt is issued to new investors, then our assumption is that the new investors will take into account the new information when determining the debt repayments. If new debt is issued to the current bank's debtholders, then we can see the issuance of new debt as the current debtholders agreeing to roll over the debt and, crucially, to reprice the debt according to the new information revealed.[8]

[8]This repricing possibility is different than other contributions in the literature, which assume that debt is rolled over with the same repayment. We believe that our repricing assumption is more consistent with unsecured wholesale debts, which are usually held by sophisticated investors.

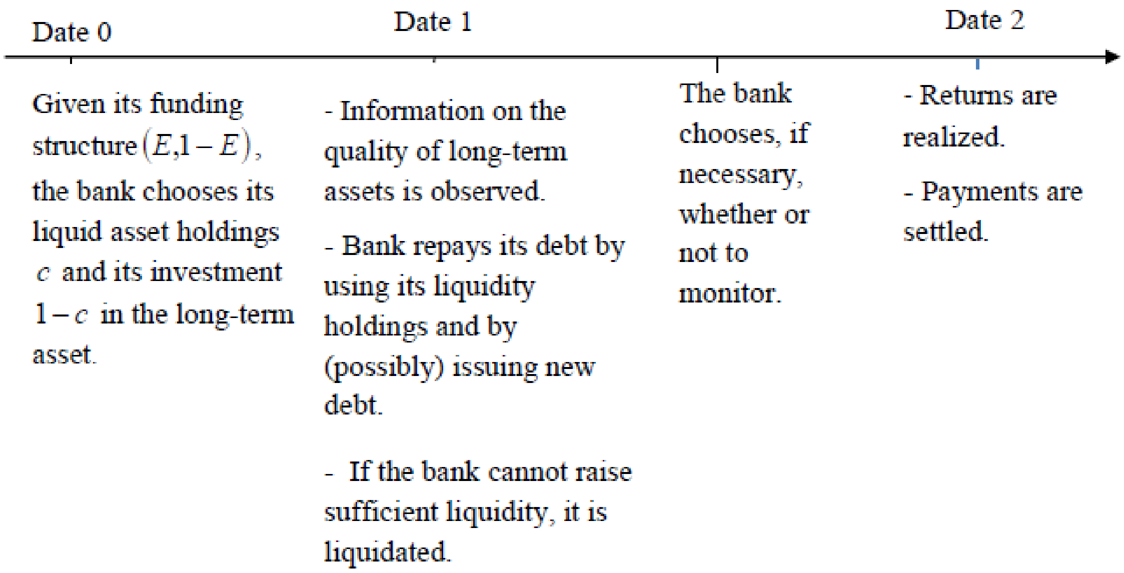



Figure 2: The timeline

We make the following assumptions on the parameters of the model.

**Assumption 1.**

$$\theta y_L \geq 1 \geq (\theta - \Delta) y_L + B$$

The main implication of Assumption 1 is that investors will lend to the bank in the case of bad news only if they are ensured that the bank will exert monitoring effort.

**Assumption 2.**

$$\ell < \theta \left( y_L - \frac{B}{\Delta} \right) < 1$$

As shown later, the middle term of Assumption 2 represents the maximum amount that the bank could borrow per unit of the long-term asset if bad news is revealed at date 1. Hence, the first inequality of this assumption has two implications. It first implies that new borrowing is a better way to raise liquidity for the bank than partial liquidation of its long-term asset. It also implies that the long-term asset is valued less in the hands of debtholders than in the hand of the bank, i.e., $\ell < \theta y_L$. The second inequality of Assumption 2 states that the amount of liquidity raised against one unit of the long-term asset in the case of bad news is lower than the amount of liquidity provided by one unit of liquid assets. Notice that this assumption ensures the role of liquid asset holdings in our setting.

**Assumption 3.**

$$\alpha\theta y_H + (1-\alpha)\,\ell > 1$$

Assumption 3 indicates that the net expected returns of the long-term asset, even if it is liquidated early on, is positive. This assumption implies that at date 0, it is still worth investing in the long-term asset, even if the bank may be closed when a liquidity shock is realized. For notional convenience, we henceforth denote the net expected return of the long-term asset as $NPV$, i.e., $NPV = \alpha\theta y_H + (1-\alpha)\theta y_L - 1$.

## 3.2 Analysis

We now analyse the bank's optimal investment decision. Our main objective is to study how many liquid assets the bank will hold on its balance sheet, and how this decision is affected by the bank's leverage. We will proceed in two steps. First, we determine the bank's funding liquidity at date 1. Then, we examine its optimal liquid asset holdings at date 0.

### 3.2.1 Bank's funding liquidity

At date 1, the bank has to repay $D_1$ to its short-term debtholders. It has $c$ units of liquidity, which implies that its liquidity needs are $D_1 - c$. The bank can raise this amount by issuing new debt repaid at date 2. Denote by $D_2$ the face value of this new debt. We now determine how much the bank can borrow at date 1 by pledging the future cash flow generated by its long-term assets.

If good information is revealed at date 1, the bank can pledge the full value of its long-term assets to investors, i.e., it can borrow up to $\theta y_H$, and there is no problem in meeting its repayment obligations.

When bad news is revealed, the incentive compatibility condition, which ensures that the bank does exert effort to monitor, is as follows:

$$\theta\left(y_L(1-c) - D_2\right) \geq (\theta - \Delta)\left(y_L(1-c) - D_2\right) + B(1-c) \tag{1}$$

Condition (1) puts a bound on the funding liquidity that the bank could get:

$$D_2 \leq \left(y_L - \frac{B}{\Delta}\right)(1-c)$$

Hence, in the case of bad news, the maximum cash flow that the bank can pledge is equal to $y_L - \frac{B}{\Delta}$. Define $\rho$ and $\rho^*$ as follows:

$$\rho = \frac{D_1 - c}{1-c} \quad \text{and} \quad \rho^* = \theta\left(y_L - \frac{B}{\Delta}\right)$$

Thus, $\rho$ is the bank's liquidity need per unit of the long-term asset, and $\rho^*$ is the maximum liquidity that the bank can raise per unit of the long-term asset. Notice that $\rho^* < \theta y_L$, i.e., the liquidity that the bank can raise from the long-term asset is strictly lower than its expected cash flow. The following lemma summarizes the bank's situation at date 1:

**Lemma 1.** *At $t = 1$:*

**(i)** *If $\rho \leq \rho^*$, the bank can always repay its short-term debt.*

**(ii)** *If $\rho > \rho^*$, the bank is liquidated when it is hit by a liquidity shock (i.e., when the bad news is revealed).*

We refer to the first situation as the one in which the bank is liquid. The second situation is referred to as the case where the bank is illiquid.

### 3.2.2 Bank's optimal precautionary liquidity holdings

In the next step, we study the bank's decision regarding the amount of liquid assets held. Given two possible situations of the bank at $t = 1$, as described in Lemma 1, we first determine how many liquid assets the bank holds in each situation. Then, we characterize the optimal liquidity holdings of the bank.

If the bank chooses $c$ so that it will be liquid at date 1, the amount of liquid assets held by the bank is determined by the following program:[9]

[9] The superscript "*li*" refers to liquid.

$$\Pi^{li} = \underset{0 \leq c \leq 1}{Max} \left\{ \alpha\theta \left[ (1-c)\, y_H - D_2^H \right] + (1-\alpha)\,\theta \left[ (1-c)\, y_L - D_2^L \right] \right\}$$

where $D_2^s$, $s = H, L$ - the face value of the new debt issued at date 1 when respectively good news or bad news is revealed - is determined as:

$$D_2^s = \frac{D_1 - c}{\theta} \text{ for all } s \tag{2}$$

subject to the break-even condition for date 0 short-term investors:

$$\alpha D_1 + (1-\alpha)\, D_1 = 1 - E \tag{3}$$

and the liquidity condition:

$$\rho \leq \rho^* \tag{4}$$

Plugging (2) and (3) into the objective function, we can rewrite the above program as follows:

$$\Pi^{li} = \underset{0 \leq c \leq 1}{Max} \left\{ NPV + E - cNPV \right\} \tag{5}$$

subject to

$$(1 - E - \rho^*) \leq c\,(1 - \rho^*) \tag{6}$$

This program makes clear the tradeoff driving the bank's liquidity holding decision. The cost of holding liquid assets is the foregone return of the long-term assets, which explains why the term "$c\,(\alpha\theta y_H + (1-\alpha)\,\theta y_L - 1)$" is deducted from the bank's expected profit. The benefit of holding liquid assets is to provide insurance against a liquidity shock at date 1, which is reflected in Constraint (6). Note that this constraint matters only if $\rho^* < 1$. One unit of liquid asset at date 0 generates one unit of liquidity at date 1, whereas the amount of liquidity raised against one unit of the long-term asset is $\rho^*$. Clearly, holding liquid assets makes sense only when $\rho^* < 1$, which is assumed to be the case in this paper, as seen in Assumption 2.

At the optimum, the bank holds an amount of liquid assets that is just sufficient to overcome a liquidity shock:

$$c^{li} = \max\left(\frac{1-E-\rho^*}{1-\rho^*}, 0\right).$$

Notice that when $E$ is high enough (i.e., $E \geq 1-\rho^*$), the bank is liquid, even though it does not hold any liquid assets. Given the optimal amount of liquid assets, the bank's expected profit when choosing to be liquid at date 1 is:

$$\Pi^{li} = NPV + E - \max\left(\frac{1-E-\rho^*}{1-\rho^*}, 0\right) NPV$$

We now turn to the amount of liquid assets that the bank holds if it chooses to be illiquid at date 1. The bank problem in this case is written as follows:[10]

$$\Pi^{illi} = \underset{0\leq c\leq 1}{Max} \left\{\alpha\theta\left[(1-c)\, y_H - D_2^H\right]\right\}$$

subject to the break-even condition of short-term investors:

$$\alpha D_1 + (1-\alpha)\left(c + (1-c)\,\ell\right) = 1 - E \tag{7}$$

and the illiquidity condition:

$$\rho > \rho^*$$

As before, $D_2^H$ is the face value of the new debt issued at date 1 in the case of good news, defined by Equation (1). Plugging (7) into the objective function and into the illiquidity condition, we obtain:

$$\Pi^{illi} = \underset{0\leq c\leq 1}{Max} \left\{\alpha\theta y_H + (1-\alpha)\,\ell - 1 + E - c\left(\alpha\theta y_H + (1-\alpha)\,\ell - 1\right)\right\}$$

subject to

$$(1-E-\rho^*) > c\,(1-\rho^*)$$

Hence, $c^{illi} = 0$ at the optimum. Since the only benefit of holding liquid assets is to provide insurance against a liquidity shock, it is intuitive that if the bank decides to be illiquid at date 1, it will not hold any liquid assets. The bank's expected profit when

[10] The superscript "*illi*" refers to illiquid.

choosing to be illiquid at date 1 is then:

$$\Pi^{illi} = \alpha\theta y_H + (1-\alpha)\,\ell - 1 + E$$

Finally, to determine the optimal liquidity holding policy of the bank, we must compare $\Pi^{li}$ and $\Pi^{illi}$. We see that the condition:

$$\Pi^{li} \geq \Pi^{illi}$$

is equivalent to

$$(1-\alpha)\,\theta y_L - (1-\alpha)\,\ell \geq \max\left(\frac{1-E-\rho^*}{1-\rho^*}, 0\right) NPV \tag{8}$$

Note that the LHS of Inequality (8) is the expected loss in value due to early liquidation of the long-term assets while the RHS represents the cost of buying insurance against liquidity risk (i.e., holding liquid assets) for the bank. Clearly, the bank chooses to be insured only if the insurance cost is lower than the loss in the value. Inequality (8) results in a condition on the bank's leverage as follows:

$$E \geq (1-\rho^*)\,\frac{\alpha\theta y_H + (1-\alpha)\,\ell - 1}{NPV} = E^* \tag{9}$$

The following proposition summarizes the characterization of the bank's optimal liquidity holding policy:

**Proposition 1.** *Precautionary liquidity holdings and leverage:*

**(i)** *When the bank is undercapitalized (i.e., $E < E^*$), it chooses to be illiquid and does not hold any liquid assets.*

**(ii)** *The bank chooses to be liquid only when it is well capitalized (i.e., $E \geq E^*$). In that case, the bank holds an amount of liquid assets equal to* $\max\left(\frac{1-E-\rho^*}{1-\rho^*}, 0\right)$ *and the liquidity coverage ratio (i.e., $\frac{c}{D_1}$) is decreasing with the bank's capital ratio.*

We represent in Figure 3 the bank's optimal liquidity holding characterized in Proposition 1. We observe that once well-capitalised, the bank will have incentives to secure some ex-ante liquidity holdings to insure itself against the liquidity shocks. This result is due to the fact that the lower the bank's capital ratio, the higher the bank's exposure to the liquidity problem is. This, in turn, leads to a higher cost of insurance (i.e., higher cost of holding liquid assets). We see clearly in Inequality (8) that the insurance cost is decreasing with the bank's capital ratio $E$. When this ratio is too low, buying insurance against a liquidity shock becomes too costly, which induces the bank to forgo insurance and gamble.[11]

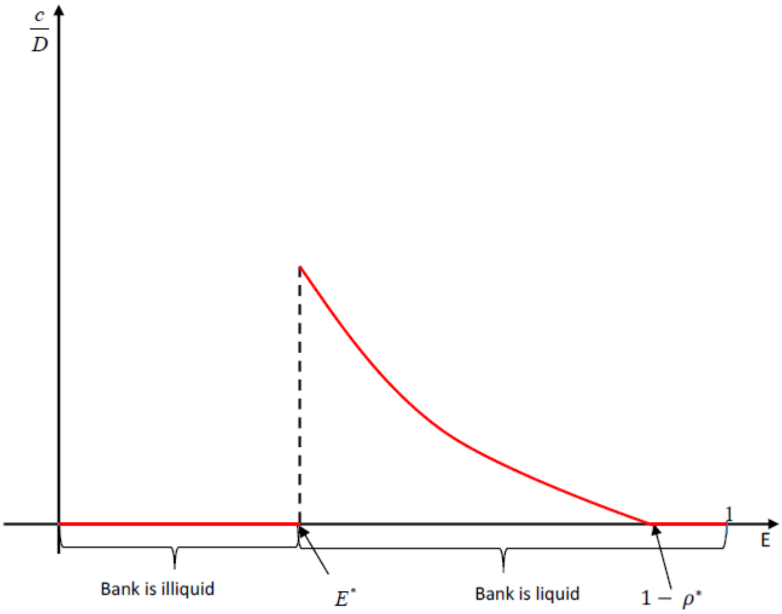



Figure 3: The bank's optimal precautionary liquidity holdings

Proposition 1 brings out a positive effect that leverage bounds may have on bank's precautionary liquidity holding incentives. In the current model, a properly designed capital requirement can perfectly do the job of improving the management of liquidity risk by banks. This result shows that any proposal concerning a liquidity requirement needs to be jointly considered with the capital regulation in order to avoid overregulation. Another interesting insight derived from Proposition 1 pertains to the impact of a decrease in the likelihood of a liquidity shock on the capital ratio threshold:

[11]The intuition behind the increasing relationship between the liquidity coverage ratio and the leverage of the bank when it is well capitalized is straightforward. Once the bank chooses to be liquid, the amount of liquid assets it holds is increasing with its exposure to liquidity risk.

**Corollary 1.** *The capital ratio threshold $E^*$is decreasing with the probability $(1-\alpha)$ that a liquidity shock happens.*

Corollary 1 states that the capital ratio threshold increases when the likelihood of the shock decreases. Put differently, the capital ratio threshold is higher for the liquidity risk that has a smaller probability of occurrence. Corollary 1 thus implies that it is much more difficult to induce banks to properly manage the tail liquidity risk.

**Credit risk vs. liquidity risk.** Our result that banks with a higher capital ratio have better incentives to manage their liquidity risk is similar to the conventional wisdom on the link between banks' capital and their credit risk-taking incentives. Nevertheless, the underlying mechanism is different.

The latter link arises when banks' creditors fail to properly price the level of credit risk taken by banks into the required debt repayments, which induces banks to engage in excessive credit risk taking. In this context, the capital level matters, since it represents the cost that banks' shareholders have to bear if their excessive behaviours lead to the closure of banks - the well-known role of *skin in the game* of banks' capital.

Note, however, that the effect of banks' capital on their incentives for liquidity risk taking in this paper does not arise because banks' creditors fall short of taking into account the liquidity risk profile of banks when determining the required interest rate. In contrast, as seen above in our model, the debt repayment is endogenously determined to make investors break-even in the expected term. Therefore, the liquidity risk taken by banks is properly priced into their borrowing rate. The impact of capital on liquidity risk taking in our setup instead comes from its property as a stable source of funding: less capital means a more unstable liability structure, and thus, higher exposure to liquidity shocks. When banks are highly undercapitalised, insuring themselves against liquidity shocks would require them to hold substantial liquid assets, which is very costly for banks. In such situations, banks will prefer not to have any insurance.

# 4 Fire-sale, liquidity crises and leverage

In order to analyse the effects of banks' leverage on fire-sale discount, and on the occurrence of liquidity crises, in this section, we enrich our previous framework by considering the existence, at date 1, of a secondary market for the long-term assets. For this purpose, we embed our previous building block of an individual bank's liquidity risk management in a setting of a continuum of banks.

## 4.1 Setup

We consider now an economy with a continuum (of mass 1) of heterogeneous banks, each indexed by $i$. Banks differ in their capital ratio with bank $i$ financed by a fraction $E_i$ of equity, and the remaining fraction being unsecured short-term debt. The repayment that bank $i$ has to make to its short-term debtholders at date $t = 1$ is denoted by $D_1^i$. We assume that $\{E_i\}_i$ is distributed according to a family of continuous distribution $F(E)$ on $[0, 1]$ with the density $f(E)$. This family of distribution is parameterised by a parameter $h$ such that an increase in the value of $h$ implies a larger mass on the left side of the distribution. Hence, the higher the value of $h$, the more highly leveraged the banking system is.

**Investment opportunities.** Each bank $i$ has access to two investment opportunities, as described in Subsection 3.1.

**Systemic liquidity shock.** At date 1, new information regarding the returns of long-term assets, as described in Subsection 3.1, becomes publicly available. We assume that the returns of long-term assets are perfectly correlated across banks, which implies that the new information will reveal the quality of the long-term assets held by all banks. Therefore, the liquidity shock in our setup is a systemic shock because it hits all of the banks simultaneously.

**Secondary market of long-term assets.** We assume that at date 1, a secondary market for long-term assets is opened, which allows banks in shortage of liquidity to sell their long-term asset holdings to raise additional liquidity. Due to some sort of asset specificity, potential purchasers of a bank's long-term assets are other banks. Moreover, purchaser banks can raise financing against the assets that they buy. Hence, following

Allen and Gale (1994, 2004, 2005), the price of long-term assets will depend on the amount of liquidity available in the banking system.

**Timing.** The extended sequence of events, which is summarised in Figure 4, is as follows. At $t = 0$, the bank decides how much to invest in each of assets. At $t = 1$, the information regarding the quality of all banks' long-term assets is revealed, and short-term debt contracts mature. If a bank's holdings of liquid assets are not enough to repay debtholders, the bank can sell part of its long-term asset holdings and can issue new debt, pledging the future payoff of the remaining fraction of its long-term assets. In the case banks cannot raise enough liquidity to repay debtholders even after selling all of their long-term assets, they are closed.[12] At date $\frac{3}{2}$, between $t = 1$ and $t = 2$, if necessary, banks decide whether to exert effort to monitor the long-term assets. At $t = 2$, long-term asset returns are realised, and all payment are settled.

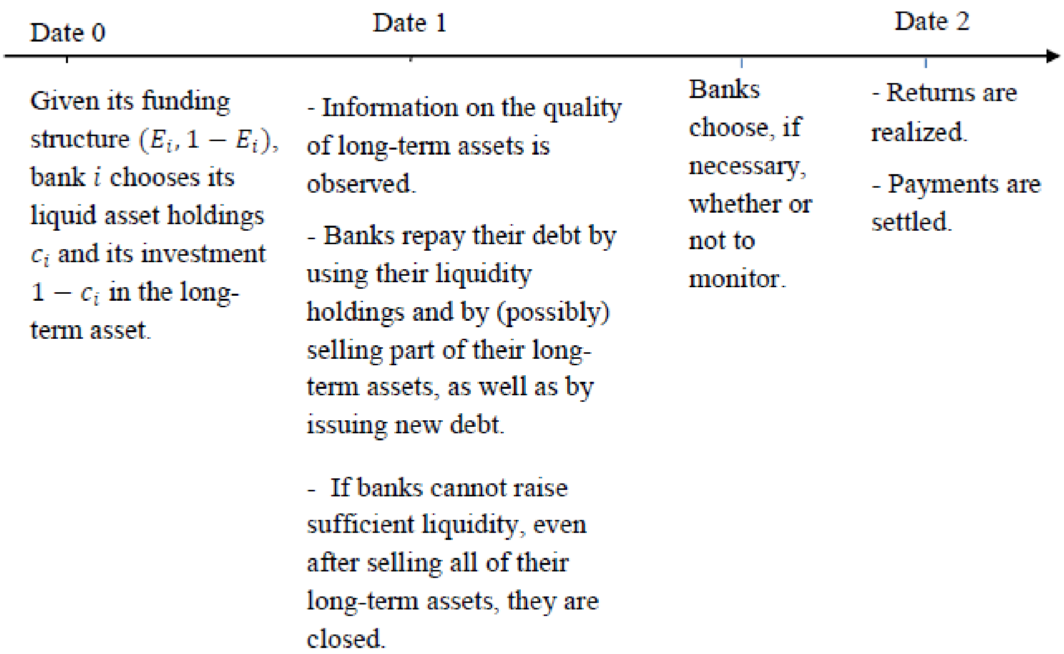



Figure 4: The extended timeline

[12] Notice the difference between asset sales in this section and asset liquidation in the previous section. Asset sales correspond to the transfer of the asset from one specialist to another who has the same ability to redeploy it. As for asset liquidation, it is equivalent to the transfer of the asset to a non-specialist who can extract a much lower surplus from the asset than a specialist.

## 4.2 Competitive equilibrium

We focus on the characteristics of competitive equilibria in which banks behave competitively in the secondary market of long-term assets. Our interest concerns how the distribution of leverage in the banking system affects the extent of the fire-sale problem and the severity of liquidity crises measured by the fraction of banks that will be closed following the materialisation of a liquidity shock.

In order to characterize the rational expectation equilibria of the present economy, we proceed as follows: we first examine the demand and supply of long-term assets at $t = 1$. Then we study the interaction between banks' ex-ante liquidity holdings and the liquidity of the secondary market for long-term assets. Finally, we characterise the competitive equilibrium, and investigate how the main properties of this equilibrium depend on the degree of leverage in the banking system.

### 4.2.1 Asset sales

As in the previous section, at date 1, if good news is realized, all banks can repay their debt. If bad news is revealed, however, banks with a liquidity shortage will have to sell their holdings of the long-term asset to raise additional liquidity. Denote by $p$ the unit price of this asset.

**Individual banks' supply.** Since the maximum funding liquidity per unit of long-term assets is $\rho^*$, banks that have to sell their long-term assets are those with $\rho$ - liquidity need per unit of long-term assets - exceeding $\rho^*$. Denote by $\beta_i$ the fraction of assets sold by bank $i$. $\beta_i$ is then determined as follows:

$$\beta_i (1 - c_i) p + (1 - c_i) (1 - \beta_i) \rho^* \geq D_1^i - c_i \tag{10}$$

In Inequality (10), the LHS is the total liquidity that bank $i$ could raise. It is the sum of the proceeds from selling the fraction $\beta_i$ and the liquidity obtained by issuing new debts against the remaining fraction $1 - \beta_i$. After simplification, we obtain:

$$\beta_i = \min \left( 1, \frac{\rho_i - \rho^*}{p - \rho^*} \right) \tag{11}$$

Observe that the funding capacity expands with asset sales if and only if the unit price $p$ is greater than $\rho^*$. We assume for now $p \geq \rho^*$, and will show later, that it is indeed the case. The extent of asset sales is decreasing with the asset's price. Bank $i$ will have to sell all of its existing long-term assets when the price $p$ falls below its liquidity demand $\rho_i$.

**Individual banks' demand.** With regard to the asset demand, banks that have $\rho$ lower than or equal to $\rho^*$ are in excess of liquidity, and thus, can buy assets. Denote by $\gamma_i$ the volume of assets that bank $i$ could buy per each unit of the long-term assets it has. Note that no banks would acquire assets at a price higher than their expected payoff. Hence, if $p > \theta y_L$, $\gamma_i$ should be equal to zero for all $i$. If $\rho^* < p < \theta y_L$, $\gamma_i$ is determined as follows:

$$(1 - c_i)(1 + \gamma_i)\rho^* - (D_i - c_i) \geq \gamma_i (1 - c_i) p \tag{12}$$

The LHS of Inequality (12) is the total liquidity available to bank $i$ for asset purchase. It consists of its spare debt capacity from existing assets, $(1 - c_i)\rho^* - (D_i - c_i)$, plus the liquidity that can be raised against assets to be acquired, $(1 - c_i)\gamma_i\rho^*$. After some arrangements, we have:

$$\gamma_i = \frac{\rho^* - \rho_i}{p - \rho^*}$$

Notice that if $p = \rho^*$, which implies that the liquidity raised against assets to be acquired is sufficient to pay for the assets, the demand for the assets is infinitely high. To summarize, the demand for long-term assets of each bank $i$ that has $\rho_i$ lower than or equal to $\rho^*$ is as follows:

$$\gamma_i(\rho_i, p) = \begin{cases} 0 & \text{if} \quad p > \theta y_L \\ \frac{\rho^* - \rho_i}{p - \rho^*} & \text{if} \quad \rho^* < p < \theta y_L \\ \text{any value between 0 and } \frac{\rho^* - \rho_i}{p - \rho^*} & \text{if} \quad p = \theta y_L \\ \infty & \text{if} \quad p = \rho^* \end{cases} \tag{13}$$

### 4.2.2 Market liquidity and ex-ante liquidity holdings

We now examine the interaction between market liquidity and banks' ex-ante liquidity holdings. We write the liquid asset holdings of each bank $i$ as follows:

$$c_i = max\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right) + \varepsilon_i \tag{14}$$

The first term in the RHS of Expression (14) is the minimum liquidity bank $i$ needs to hold to overcome the liquidity shock at date 1. A strictly positive $\varepsilon_i$ is equivalent to a liquid bank holding excess liquidity. A strictly negative $\varepsilon_i$ implies that bank $i$ is illiquid. Given that $0 \leq c_i \leq 1$ for all $i$, $\varepsilon_i$ must satisfy the following conditions:

$$min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right) \leq \varepsilon_i \leq min\left(\frac{E_i}{1-\rho^*}, 1\right) \quad \text{for} \quad \text{all} \quad i \tag{15}$$

Note that as shown in Section 3.2, when banks hold liquidity for precautionary motive only, $\varepsilon_i$ is equal to zero for any banks $i$ with capital ratio not lower than $E^*$ and to $-\frac{1-\rho^*-E_i}{1-\rho^*}$ for other banks.

We now study how the possibility of buying and selling long-term assets affects banks' incentives to hold liquidity ex-ante. Using Expression (14), we could express, in terms of $\varepsilon_i$, the volume of long-term assets bought by any bank $i$ with excess liquidity (i.e. $\rho_i \leq \rho^*$) as follows:

$$(1-c_i)\gamma_i = \frac{\varepsilon_i(1-\rho^*)}{p-\rho^*} - min\left(\frac{1-\rho^*-E_i}{p-\rho^*}, 0\right) \quad \text{for} \quad \rho^* < p < \theta y_L$$

In relation to the volume of assets sold by bank $i$ with liquidity shortage (i.e., $\rho_i > \rho^*$), it could be written as:

$$(1-c_i)\beta_i = \begin{cases} -\frac{\varepsilon_i(1-\rho^*)}{p-\rho^*} + min\left(\frac{1-\rho^*-E_i}{p-\rho^*}, 0\right) & \text{if} \quad \rho^* < \rho_i < p \\ 1-\varepsilon_i - max\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right) & \text{if} \quad \rho_i \geq p \end{cases} \tag{16}$$

Hence, the problem that determines the optimal liquidity holdings of a bank $i$ can be written, in term of $\varepsilon_i$, as follows:[13]

**Program** $\wp$

$$\begin{aligned}\underset{\varepsilon_i}{\text{Max}} \Bigg\{ & NPV + E_i - \left( max\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right) + \varepsilon_i \right) NPV \\ & +(1-\alpha)(\theta y_L - p)\left[\frac{\varepsilon_i(1-\rho^*)}{p-\rho^*} - min\left(\frac{1-\rho^*-E_i}{p-\rho^*}, 0\right)\right]\mathbb{1}_{\rho_i<p} \\ & -(1-\alpha)(\theta y_L - p)\left(1-\varepsilon_i - max\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)\right)\mathbb{1}_{\rho_i\geq p}\Bigg\} \end{aligned} \tag{17}$$

subject to $\varepsilon_i$ satisfying Conditions (15).

Note that at date 1, following the realisation of the liquidity shock, a bank $i$ can be in either one of the three situations: It either has to sell all of its long-term assets and be closed or can survives the liquidity shock after selling a fraction of its long-term assets or is liquid and can buy the long-term assets sold by illiquid banks. The fourth term in the bracket of Expression (17) represents either the additional profit bank $i$ could get if it is the buyer in the market or the loss it must incur if it has to sell a fraction of its long-term assets to overcome the liquidity shock. The last term in the bracket represents the loss that it has to incur if it is closed at date 1. The absolute value of these terms are strictly positive only when the price of long-term assets is strictly below their fundamental value $\theta y_L$.

Define $\delta$ as follows:

$$\delta = \frac{(1-\alpha)(1-\rho^*)(\theta y_L - \rho^*)}{NPV + (1-\alpha)(1-\rho^*)} \tag{18}$$

The following lemma summarises the impact of market liquidity on banks' incentives to hold liquidity ex-ante.

**Lemma 2.** *Market liquidity and ex-ante liquidity holdings:*

**(i)** *If* $\rho^* < p < \rho^* + \delta$*:*

[13] For the detailed derivation of this problem, see the Appendix.

$$\varepsilon_i = min\left(\frac{E_i}{1-\rho^*}, 1\right) \mathbb{1}_{\rho_i \leq p} + min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right) \mathbb{1}_{\rho_i > p}$$

**(ii)** *If* $\rho^* + \delta < p \leq \theta y_L$*:*

$$\varepsilon_i = min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)$$

The first part of Lemma 2 states that if expecting the price to be low (i.e. below $\rho^* + \delta$), banks that are not too highly leveraged will hold excess liquidity.[14] The reason is that when price is low, the gains from acquiring assets cheap is large. This gives banks higher incentives to hold liquid assets so that they could survive the liquidity shock and have resources to take advantage of fire sales. It is the so-called *speculative motive of liquidity holdings* described in the literature (e.g. Acharya et al. (2011a)). The second part of the lemma is its counterpart: if expecting the price to be high enough, banks will have no incentives to secure some sources of liquidity ex-ante. That is because when price is high, banks could rely on the market liquidity of long-term assets to overcome the liquidity shock and save on the cost of holding liquidity. Hence, higher (lower) market liquidity does not necessarily imply a reduction (an increase) of the severity of the liquidity problem if taking into account its ex-ante effect on banks' liquidity holdings.

### 4.2.3 Characterisation of competitive equilibrium

We are now equipped to examine the existence and the main features of the competitive equilibrium. Our focus is to analyse how the extent of the fire-sale problem, and the fraction of banks that would be closed following the crystallisation of a liquidity shock vary with the leverage of the banking system measured by $h$. For the purpose of this analysis, from now on we will indicate the banks' leverage distribution as $[F(E, h), f(E, h)]$ in order to clarify that the shape of the distribution depends on parameter $h$.

*Definition of the ex-ante competitive equilibrium:* A competitive equilibrium in our setup is (1) a set of banks' liquidity holdings $\{c_i^*\}_{i \in [0,1]}$; and (2) the equilibrium price $p^e$

[14] In our setup, these banks will invest all of their funding in liquid assets. This extreme result is due to our assumptions about the constant return to scale and the divisibility features of long-term assets, which implies that banks' expected profit is linear in their liquidity holdings.

of long-term the assets at date 1 following the revelation of bad news such that:

**(1)** $c_i^*$ is the optimal amount of liquid assets that each bank $i$ holds, given $p^e$.

**(2)** $p^e$ is the equilibrium price induced by the choices $\{c_i^*\}_{i\in[0,1]}$.

We state the first property of the competitive equilibrium in the following proposition:

**Proposition 2.** *Only a competitive equilibrium where $p^e \leq \rho^* + \delta < \theta y_L$ can exist.*

The intuition underlying Proposition 2 comes directly from Lemma 2. In fact, as highlighted in this lemma, if banks expect the price to be higher than $\rho^* + \delta$, no banks would have incentives to hoard liquidity ex-ante. As a consequence, there is no liquidity available ex-post to endorse this price. In other words, such a price is not supportable in the equilibrium.

Proposition 2 points out that the rational expectation equilibrium in our economy always features a fire sale. This attribute pertains to the *systemic* nature of our liquidity shock. In a setting with *idiosyncratic* liquidity shocks, an equilibrium where the price is equal to the fundamental value can exist if the fraction of banks that are hit by the shock is small enough.[15] However, it cannot be the case when the shock is systemic for two reasons. First, a systemic shock, once crystallised, will hit all banks at the same time. Moreover, no banks would have incentives to hold excess liquidity ex-ante to absorb assets if they expect those assets to be traded at their fundamental value.

A direct implication of Proposition 2 is that in the equilibrium, the price is either equal to $\rho^* + \delta$ or strictly lower. The following proposition describes the main characteristics of these two *possible* equilibria.

**Proposition 3.** *Main properties of possible competitive equilibria:*

1. *The equilibrium where the price is equal to $\rho^* + \delta$, if it exists, has the following features:*

   - *Banks with a capital ratio lower than the cutoff level $1 - \rho^* - \delta$ hold zero liquidity and will be closed at date 1 following the realisation of the liquidity shock.*

[15] See, for example, Acharya et al. (2011a).

- *Banks with a capital ratio greater than or equal to* $1-\rho^*-\delta$ *are indifferent to any liquidity holdings between max* $\left(\frac{1-\rho^*-E_i}{1-\rho^*},0\right)$ *and* $1$ *and will survive the shock.*

2. *The equilibrium in which the price is strictly lower than* $\rho^*+\delta$*, if it exists, is also characterised by a cutoff capital ratio* $\overline{E}$ *such that:*

   - *Banks with a capital ratio lower than* $\overline{E}$ *hold zero liquidity and will be closed at date* $1$ *following the realisation of the liquidity shock.*
   - *Banks with a capital ratio greater than or equal to* $\overline{E}$ *invest all funds in liquid assets and will survive the shock.*

   *The cutoff level* $\overline{E}$ *and the price* $p^e$ *in this equilibrium are determined by the following conditions:*

$$\frac{\overline{E}}{p^e-\rho^*}+1-\frac{NPV}{(1-\alpha)(\theta y_L-p^e)}=0 \tag{19}$$

$$\int_{\overline{E}}^{1} Ef(E,h)dE = p^e\int_0^{\overline{E}} f(E,h)dE \tag{20}$$

$$p^e<\rho^*+\delta \tag{21}$$

That said, in the equilibrium, the banks that will fail at date 1 are banks with a capital ratio below a cutoff level denoted by $\hat{E}^e$. The precise value of this threshold depends on which of the two possible equilibria described in Proposition 3 will prevail. It is worth elaborating more on two Equations (19) - (20) that determine together the cut-off level and the equilibrium price in the second possible equilibrium. Equation (20) is the equalisation of, on the LHS, the total liquidity available to absorb the long-term assets at date 1 and, on the RHS, the total market value of all assets offered for sale. Equation (19) determines the marginal banks that would fail at date 1. It is derived by equalising the expected profit they would get if their choice of liquidity holdings allows them to survive the liquidity a shock and if it does not.

Notice that the solutions to the system of the two Equations (19) - (20) depend on $h$. Denote them by $\overline{E}(h)$ and $p(\overline{E}(h),h)$. The prevailing equilibrium will be the one with a price strictly lower than $\rho^* + \delta$ if $p(\overline{E}(h), h)$ satisfies Condition (21). Otherwise, the prevailing equilibrium is the one with a price equal to $\rho^* + \delta$. It is thus necessary, in order to establish the prevailing equilibrium, to examine how $p(\overline{E}(h), h)$ varies with $h$. We present the main results in the below proposition.

**Proposition 4.** *Prevailing equilibrium*

**(i)** *There exists a unique couple* $\left[\overline{E}(h), p(\overline{E}(h), h)\right]$ *that solves the system of the two Equations (19) - (20). We have:*

- $\overline{E}(h)$ *is a decreasing function of* $h$.
- $p(\overline{E}(h), h)$ *is decreasing with respect to both* $\overline{E}$ *and* $h$.

**(ii)** *There exists a unique threshold* $\hat{h}$ *defined by* $p(\overline{E}(\hat{h}), \hat{h}) = \rho^* + \delta$ *such that:*

- *For* $h \leq \hat{h}$, *the prevailing equilibrium is the one in which the equilibrium price* $p^e$ *equals* $\rho^* + \delta$.
- *For* $h > \hat{h}$, *the prevailing equilibrium is the one where the equilibrium price* $p^e$ *equals* $p(\overline{E}(h), h)$, *which is strictly lower than* $\rho^* + \delta$.

The cutoff capital ratio and the unit price of the long-term assets in the equilibrium can therefore be summarised respectively as follows:

$$\hat{E}^e(h) = \begin{cases} 1 - \rho^* - \delta & \text{if} \quad h \leq \hat{h} \\ \overline{E}(h) & \text{if} \quad h > \hat{h} \end{cases} \tag{22}$$

and

$$p^e\left(\hat{E}^e(h), h\right) = \begin{cases} \rho^* + \delta & \text{if} \quad h \leq \hat{h} \\ p(\overline{E}(h), h) & \text{if} \quad h > \hat{h} \end{cases} \tag{23}$$

Since the capital ratio in the banking system is distributed according to the distribution $F(E, h)$, the fraction of the banks that fail at date 1 in the equilibrium can be computed as $F(\hat{E}^e(h), h)$.

### 4.3 Fire-sale problem and severity of liquidity crises

We now study the impact of the leverage distribution in the banking system (i.e., the impact of a change in $h$) on the extent of the fire-sale problem and the severity of liquidity crises measured by the fraction of banks that will be closed following the crystallisation of the liquidity shock. We compute the total derivative of $p^e(\hat{E}^e(h), h)$ and $F\left(\hat{E}^e(h), h\right)$ with respect to $h$ as follows:

$$\frac{dp^e(\hat{E}^e(h), h)}{dh} = \underbrace{\overbrace{\frac{\partial p^e(\hat{E}^e(h), h)}{\partial h}}^{\leq 0}}_{\text{Effect of precautionary motive}} + \underbrace{\overbrace{\frac{\partial p^e(\hat{E}^e(h), h)}{\partial \hat{E}^e}}^{<0}\overbrace{\frac{\partial \hat{E}^e(h)}{\partial h}}^{\leq 0}}_{\text{Effect of speculative motive}} \tag{24}$$

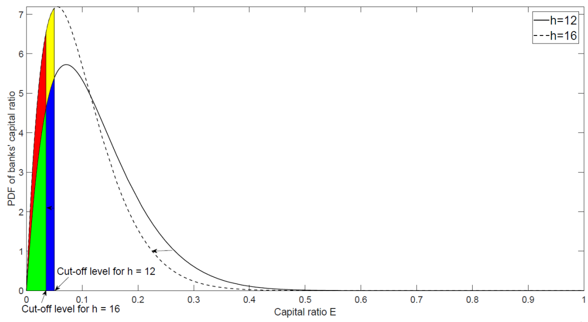
and

$$\frac{dF(\hat{E}^e, h)}{dh} = \underbrace{\overbrace{\frac{\partial F(\hat{E}^e(h), h)}{\partial h}}^{\geq 0}}_{\text{Effect of precautionary motive}} + \underbrace{\overbrace{\frac{\partial F(\hat{E}^e(h), h)}{\partial \hat{E}^e}}^{\geq 0}\overbrace{\frac{\partial \hat{E}^e(h)}{\partial h}}^{\leq 0}}_{\text{Effect of speculative motive}} \tag{25}$$

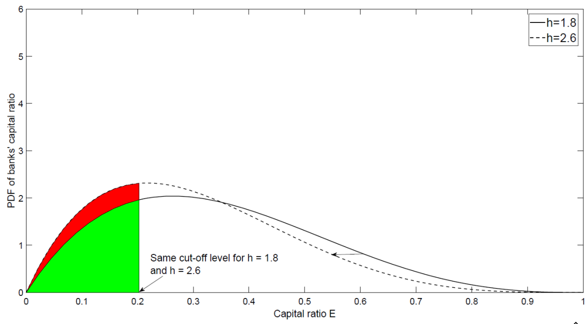
We can see that a change in $h$ has two effects on these two variables. The first effect, represented by the first terms on the RHS of Expressions (24) - (25), works through the impact of $h$ on the shape of the banks' leverage distribution $F(E, h)$. The second effect, accounted for by the second terms on the RHS of Expressions (24) - (25), comes from the impact of $h$ on the identity of the marginal banks that will fail (i.e., the impact on $\hat{E}^e$). Basically, an increase in $h$ means that there are more banks with low capital ratio. The immediate effect of an increase in the number of highly leveraged banks is to increase the pool of banks that *would* choose to be illiquid, and thus, weakly decrease the expected price. This decrease in the expected price increases the gains from buying assets cheaply, which in turn leads to an interesting second effect; that is, the increase in banks' incentives to hold liquidity, or formally, the cutoff capital ratio decreases. As a result, the pool of the banks that *will* choose to hold enough liquidity at date 0 in order to survive the liquidity shock and have enough resources to take advantage of the fire sale is larger than what would be without moving the capital ratio threshold.[16]

[16] Formally, looking at the two Equations (19) - (20), we see that a change in $h$ first induces a change in $p^e$, which in turn leads to a change in $\overline{E}$. We can show that the partial derivative of $\overline{E}$ with respect

Figure 5 illustrates, in a numerical example, the impact of a shift in the banks' leverage distribution. When the probability density function (pdf) of banks' capital ratio is the solid line, the fraction of the banks that will fail at date 1 is represented by the blue and green areas in Panel A, and by the green area in Panel B.




(a) Panel A: Impact of a change in $h$ when $h > \hat{h}$




(b) Panel B: Impact of a change in $h$ when $h < \hat{h}$

Figure 5: **Impact of a change in $h$**

The figure shows the numerical results for the case in which the capital ratio of the banks in the system is distributed according to the beta distribution, parametrized by two positive parameters, $a$ and $b$. We fix $a$ at 2 and vary the value of $h = b - a$. The numerical values for other parameters are as follows: $\alpha = 0.6, \theta = 0.7, y_H = 1.8, y_L = 1.5, \Delta = 0.4$ and $B = 0.3$.

- In Panel A, when the banks' leverage distribution shifts from the solid line to the dotted line, as explained above, there are two countervailing effects at play. First, if we keep $\hat{E}^e$ fixed, this shift of the distribution increases the number of banks that would be illiquid, which is now represented by a combination of 4 areas, namely blue, green, yellow, and red. This drives down the expected price and moves the threshold $\hat{E}^e$ to the left, since the increase in the expected gains from buying assets cheaply induces more banks to choose to hold liquidity. Hence, in the new equilibrium, the fraction of banks that will be closed at date 1 is composed of the green and red areas.

- In Panel B, the second effect is mute. The reason is that when the banking system is well capitalised (i.e., $h$ is small), the spare capacity of the liquidity holdings among well-capitalised banks is still high. Therefore, a small shift in the leverage distribution does not lead to a decrease in the equilibrium price. Formally, as stated

to $p^e$ are non negative.

in Proposition 3, when $h$ is still low (i.e., $h < \hat{h}$), banks with a capital ratio greater than $1 - \rho^* - \delta$ are indifferent to any liquidity holdings between $max\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)$ and 1. Following a small shift to the left of the banks' leverage distribution, these banks can increase their liquidity holdings, and thus, the pre-shift equilibrium price can still be supportable.

We refer to the first effect as the *effect of the precautionary motive* for liquidity holdings, since it reflects the pure impact of banks' leverage on their incentives to hold liquidity, as analysed in Section 3.2. The second effect is referred to as the *effect of the speculative motive* for liquidity holdings, since it expresses the impact of the changes in the banks' leverage distribution on the gains from buying assets. Therefore, when the banking system becomes more highly leveraged, the precautionary motive for liquidity holdings results in a weak decrease in the equilibrium price and an increase in the fraction of the banks that fail following the materialisation of the liquidity shock, while the speculative motive for liquidity holdings has the opposite effect. The overall impact thus depends on which one of the two effects is stronger.

From Expression (23) and Part 1 of Proposition 4, we see that the effect of the speculative motive on the equilibrium price is never stronger than the impact of the precautionary motive. As a result, the equilibrium price of long-term assets is weakly decreasing with $h$. This is intuitive, since banks will never increase their liquidity holdings to the extent that it saturates the increase in the gains from buying assets. Otherwise they will not be able to *actually* encash any additional profits.

**Corollary 2.** *The equilibrium price of long-term assets is weakly decreasing with the degree of leverage in the banking system.*

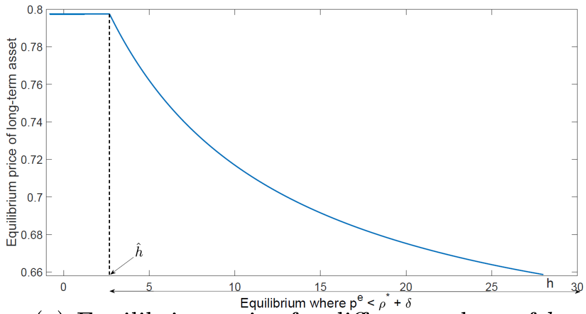
**Numerical analysis.** In relation to the overall impact of a change in the banks' leverage distribution on the fraction of the banks that will fail at date 1, unfortunately it is not possible to analytically verify the sign of the total effect. We thus complement our analysis with some numerical results. Our baseline parameter values are as follows: good news is revealed with probability equal to 0.6. In that case, the long-term assets yield a payoff equal to 1.8 with probability $\theta$ equal to 0.7. If bad news arrives, the successful cash flow of the long-term assets is reduced to 1.5 and, without banks' monitoring effort, the probability of success is reduced by $\Delta = 0.4$. Banks' private benefits in the case of shirking are assumed to be equal to 0.3. The capital ratio of banks in the system is distributed according to the beta distribution, parametrized by two positive parameters, $a$ and $b$.

In order to illustrate the effect of the banks' leverage distribution on the extent of the fire-sale problem and the severity of liquidity crises, we numerically solve the system of the two Equations (19) - (20) while varying the difference between the two shape parameters of the beta distribution. Precisely, we fix $a$ at 2 and vary the value of $h = b - a$. Notice that when $a$ is fixed, the higher the value of $h$, the bigger the mass of the distribution on the left is. This therefore allows us to interpret an increase in the value of $h$ as a more highly leveraged banking system.

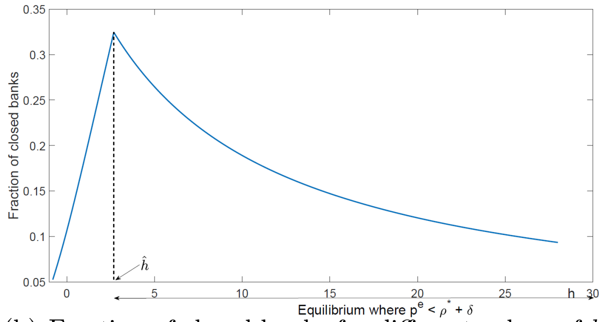
We pin down the competitive equilibrium corresponding to each value of $h$ by checking whether the resulting price from solving the two Equation (19) - (20) satisfies Condition (21). If so, the corresponding equilibrium is the one characterised in Part 2 of Proposition 3. Otherwise, the corresponding equilibrium is the one in which the equilibrium price equals $\rho^* + \delta$.

We show in Figure 6(a) the equilibrium price, and in Figure 6(b), the fraction of the banks that will be closed following the crystallisation of the liquidity shock. Consistent with Corollary 2, the equilibrium price is non increasing with $h$ and is strictly decreasing when $h$ is high enough. *Interestingly, we see that the proportion of the banks that will fail when the liquidity shock is materialised is not monotonic with respect to the degree of leverage in the banking system.*

Two main insights emerge from Figure 6. First, the effects of the changes in the capitalisation of the banking system on the severity of liquidity crises depend on the initial level of the system's capitalisation. Improving the banking system's capitalisation is beneficial, except when the system is poorly capitalised. Second, a severe fire-sale problem and a high proportion of bank failures in the system do not necessarily happen together.



(a) Equilibrium price for different values of $h$



(b) Fraction of closed banks for different values of $h$

Figure 6: Characteristics of the competitive equilibrium

# 5 Conclusion

This paper develops a model of banks' liquidity management to examine the link between banks' capitalisation and their liquidity-risk taking as well as the extent of the fire-sale problems and liquidity crises. We find that banks have incentives to hold an adequate amount of liquidity to protect themselves against future liquidity shocks only if they are well capitalized. We also find that from the system's perspective, when the speculative of liquidity holdings is taken into account, the fire-sale discount is weakly increasing with the degree of leverage in the banking system. However, the proportion of banks that will fail when a liquidity shock is materialised is not monotonic with respect to the capitalisation of the banking system. Improving the banking system's capitalisation is beneficial, except when the system is poorly capitalised.

We believe that the present framework provides a useful springboard for future research that helps deepen our understanding of the impact of banks' leverage on their incentives for liquidity management. One promising extension would be to endogenise the banks' choice of leverage. This would allow us to analyse, for example, the effects of banks' capital on their effort to reduce the likelihood of a liquidity shock. Another interesting extension would be to take into account the role of long-term debt and ask whether holding liquid assets and funding by long-term debt are perfect substitutes from a liquidity risk perspective.

# Appendix

**Derivation of Program $\wp$.** Note that at date 1, following the realisation of the liquidity shock, a bank $i$ either has to sell all of its long-term assets and be closed or can survives the liquidity shock after selling a fraction of its long-term assets or is liquid and can buy the long-term assets sold by illiquid banks. Its expected profit can therefore be written as follows:

$$\Pi_i = NPV + E_i - c_i NPV + (1-\alpha)(1-c_i)\gamma_i(\theta y_L - p)\mathbb{1}_{\rho_i \leq \rho^*} \\ - (1-\alpha)(1-c_i)\beta_i(\theta y_L - p)\mathbb{1}_{\rho_i > \rho^*} \quad \text{(A1)}$$

Therefore, the problem that determines the optimal liquidity holdings of bank $i$ is as follows:

$$\underset{c_i \in [0,1]}{\text{Max}}\, \Pi_i \quad \text{(A2)}$$

subject to

$$\alpha D_1^i + (1-\alpha)D_1^i \mathbb{1}_{\rho_i \leq \rho^*} + (1-\alpha)min\left[D_1^i, (1-c_i)\beta_i p + (1-c_i)\left(1-\beta_i\right)\rho^* + c_i)\right]\mathbb{1}_{\rho_i > \rho^*} \\ = 1 - E_i \quad \text{(A3)}$$

$$\frac{D_1^i - c_i}{1 - c_i} = \rho_i \quad \text{(A4)}$$

$$\beta_i = \min\left(1, \frac{\rho_i - \rho^*}{p - \rho^*}\right) \quad \text{(A5)}$$

$$\gamma_i = \frac{\rho^* - \rho_i}{p - \rho^*} \quad \text{for} \quad \rho^* < p < \theta y_L \quad \text{(A6)}$$

To express the above program in term of $\varepsilon_i$, let us first rewrite the fourth term in the RHS of Expression (A1). We have:

$$(1-c_i)\gamma_i = \frac{(1-c_i)\rho^* - D_1^i + c_i}{p - \rho^*} = \frac{(1-c_i)\rho^* - 1 + E_i + c_i}{p - \rho^*} \quad \text{(A7)}$$

where the second equality comes from Condition (A3) under the case $\rho_i \leq \rho^*$. Replacing $c_i$ as defined in Expression (14) into Expression (A7), we obtain:

$$(1-c_i)\gamma_i = \frac{\varepsilon_i(1-\rho^*)}{p-\rho^*} - min\left(\frac{1-\rho^*-E_i}{p-\rho^*}, 0\right) \quad \text{for} \quad \rho^* < p < \theta y_L \tag{A8}$$

We proceed similarly with the last term in the RHS of Expression (A1) and obtain:

$$(1-c_i)\beta_i = \begin{cases} -\frac{\varepsilon_i(1-\rho^*)}{p-\rho^*} + min\left(\frac{1-\rho^*-E_i}{p-\rho^*}, 0\right) & \text{if} \quad \rho^* < \rho_i < p \\ 1-\varepsilon_i - max\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right) & \text{if} \quad \rho_i \geq p \end{cases} \tag{A9}$$

Then by plugging Expressions (A8) - (A9) into Expression (A1), we get the objective function of **Program** $\wp$.

**Proof of Lemma 2.** To determine the optimal liquidity holdings, let us compute the first derivative of Expression (17) that is equal as follows:

$$-NPV + \frac{(1-\alpha)(1-\rho^*)(\theta y_L - p)}{p-\rho^*}\mathbb{1}_{\rho_i<p} + (1-\alpha)(\theta y_L - p)\mathbb{1}_{\rho_i \geq p} \tag{A10}$$

It is obvious that for banks with $\rho_i \geq p$, Expression (A10) is strictly negative. For banks with $\rho_i < p$, after some arrangements, we see that Expression (A10) is strictly positive if and only if the following condition is satisfied:

$$p < \rho^* + \frac{(1-\alpha)(1-\rho^*)(\theta y_L - \rho^*)}{NPV + (1-\alpha)(1-\rho^*)} = \rho^* + \delta \tag{A11}$$

Hence, as stated in Lemma 2, we have:

- If $p > \rho^* + \delta$, Expression (17) is strictly decreasing with $\varepsilon_i$ for all $i$, which means that at the optimum $\varepsilon_i = min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)$.
- If $p < \rho^* + \delta$, Expression (17) is strictly increasing with $\varepsilon_i$ for banks with $\rho_i < p$, and strictly decreasing with $\varepsilon_i$ for banks with $\rho_i \geq p$. Therefore, at the optimum:

$$\varepsilon_i = min\left(\frac{E_i}{1-\rho^*}, 1\right)\mathbb{1}_{\rho_i<p} + min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)\mathbb{1}_{\rho_i \geq p} \tag{A12}$$

**Proof of Proposition 2.** Note that from Lemma 2, we see that if banks expect $p > \rho^* + \delta$, they will all hold zero liquidity. Hence, there is no liquidity available ex-post to support this price. In other words, an equilibrium where $p > \rho^* + \delta$ cannot exist.

**Proof of Proposition 3.**

- We start first with the characterisation of the equilibrium in which the price is strictly lower than $\rho^* + \delta$.

  From Lemma 2, if expecting price to be strictly lower than $\rho^* + \delta$, we know that the banks' optimal liquidity holdings are as follows:

$$\varepsilon_i = min\left(\frac{E_i}{1-\rho^*}, 1\right)\mathbb{1}_{\rho_i \leq p} + min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)\mathbb{1}_{\rho_i > p}$$

  Therefore, we need to determine when a bank $i$ will choose their liquidity holding such that $\rho_i \leq p$ or $\rho_i > p$ by comparing its expected profits in two cases. If bank $i$ chooses to be illiquid, its expected profit is as follows:

$$\Pi_i^{illi} = NPV + E_i - (1-\alpha)(\theta y_L - p) \tag{A13}$$

  If bank $i$ chooses to be liquid, its expected profit is as follows:

$$\Pi_i^{illi} = E_i + (1-\alpha)(\theta y_L - p)\frac{E_i}{p-\rho^*} \tag{A14}$$

  Hence, the cutoff capital ratio $\overline{E}$ is determined by the following conditions:

$$\overline{E} + (1-\alpha)(\theta y_L - p)\frac{\overline{E}}{p-\rho^*} = NPV + \overline{E} - (1-\alpha)(\theta y_L - p) \tag{A15}$$

  which is equivalent to:

$$\frac{\overline{E}}{p-\rho^*} + 1 = \frac{NPV}{(1-\alpha)(\theta y_L - p)} \tag{A16}$$

  Note that all banks with $E_i \geq \overline{E}$ will choose to invest all of their funds in the liquid asset, which implies that the spare liquidity of each bank $i$ is $E_i$. Therefore, the total spare liquidity is equal to $\int_{\overline{E}}^{1} Ef(E,h)dE$. Since all banks with $E_i < \overline{E}$ will hold zero liquidity and will sell all of their long-term assets if the liquidity shock is realised, the total supply of the long-term assets in the secondary market is $\int_0^{\overline{E}} f(E,h)dE$. Hence, the market clearing condition implies that the equilibrium price is determined by the following equation:

$$\int_{\overline{E}}^{1} Ef(E,h)dE = p^e \int_0^{\overline{E}} f(E,h)dE \tag{A17}$$

To summarise, in the equilibrium where $p < \rho^* + \delta$, the cutoff capital level and the equilibrium price are jointly determined by the following equations:

$$\frac{\overline{E}}{p-\rho^*} + 1 = \frac{NPV}{(1-\alpha)(\theta y_L - p)} \tag{A18}$$

$$\int_{\overline{E}}^{1} Ef(E,h)dE = p^e \int_0^{\overline{E}} f(E,h)dE \tag{A19}$$

- We now characterise the equilibrium in which the price is equal to $\rho^* + \delta$.

From Program$\wp$, we see that banks with $\rho_i > p$ will hold zero liquidity, and, consequently, will have to sell all of their long-term assets. Their profits will thus be equal:

$$\Pi_i^{illi} = NPV + E_i - (1-\alpha)(\theta y_L - \rho^* - \delta) \tag{A20}$$

For banks that have $\rho_i < p$, we also see that they will be indifferent to any liquidity holdings between $max\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)$ and 1. Their expected profit is as follows:

$$\Pi_i^{li} = E_i + \frac{E_i}{1-\rho^*} NPV \tag{A21}$$

Note that the condition $\rho_i > p = \rho^* + \delta$ implies that $\frac{D_1^i - c_i}{1-c_i} > \rho^* + \delta$. Since a bank $i$ that has $\rho_i > p$ will be closed at date 1, $D_1^i$ is determined as follows

$$\alpha D_1^i + (1-\alpha)p = 1 - E_i \tag{A22}$$

Therefore, condition $\rho_i > p = \rho^* + \delta$ is equivalent to $E_i < 1 - \rho^* - \delta$. Notice also that if $E_i < 1 - \rho^* - \delta$, then we have

$$\Pi_i^{li} < \Pi_i^{illi} \tag{A23}$$

which means that banks with capital ratio lower than $1-\rho^*-\delta$ will indeed prefer to hold zero liquidity. They will thus be closed at date 1 following the realisation of the liquidity shock.

**Proof of Proposition 4.**

- *First, we will show that the system of the two Equations (19) - (20) has unique solutions.*

  Indeed, from Equation (20), we can derive $p^e$ as a function of $\overline{E}$ and $h$ as follows:

$$p^e = \frac{\int_{\overline{E}}^{1} Ef(E,h)dE}{\int_0^{\overline{E}} f(E,h)dE} \tag{A24}$$

  Computing the partial derivative of $p^e$ with respect to $\overline{E}$, we have:

$$\frac{\partial p^e}{\partial \overline{E}} = -\frac{f(\overline{E},h)\left[\overline{E}\int_0^{\overline{E}} f(E,h)dE + \int_{\overline{E}}^{1} Ef(E,h)dE\right]}{\left(\int_0^{\overline{E}} f(E,h)dE\right)^2} < 0 \tag{A25}$$

  Hence, $p^e$ is a decreasing function of $\overline{E}$. Define $\overline{E}_1$ and $\overline{E}_2$ as follows:

$$p^e\left(\overline{E}_1\right) = \theta y_L \quad \text{and} \quad p^e\left(\overline{E}_2\right) = \rho^* \tag{A26}$$

  Since $p^e$ is a decreasing function of $\overline{E}$, we have $\overline{E}_1 < \overline{E}_2$. Given that the natural boundaries for $p^e$ are $\rho^*$ and $\theta y_L$ (i.e, $\rho^* \le p^e \le \theta y_L$), we are only interested in the solution where $\overline{E}_1 \le \overline{E} \le \overline{E}_2$.

  Define

$$g(\overline{E}) = \frac{\overline{E}}{p^e - \rho^*} \tag{A27}$$

  and $G(\overline{E})$ as the left-hand side of Equation(19), i.e,

$$G(\overline{E}) = g\left(\overline{E}\right) + 1 - \frac{NPV}{(1-\alpha)(\theta y_L - p^e)} \tag{A28}$$

where $p^e$ is computed using Expression (A24). Hence, to show that the system of the two Equations (19) - (20) has unique solutions, we need to show that the following equation has unique solution in the interval $\left[\overline{E}_1, \overline{E}_2\right]$:

$$G\left(\overline{E}\right) = 0 \tag{A29}$$

We have:

$$\frac{\partial g(\overline{E})}{\partial \overline{E}} = \frac{p^e - \rho^* - \overline{E}\frac{\partial p^e}{\partial \overline{E}}}{\left(p^e - \rho^*\right)^2} \tag{A30}$$

Since $\frac{\partial p^e}{\partial \overline{E}} < 0$ and $p^e > \rho^*$ for $\overline{E}_1 \leq \overline{E} \leq \overline{E}_2$, we have:

$$\frac{\partial g(\overline{E})}{\partial \overline{E}} > 0 \quad \forall \overline{E}_1 \leq \overline{E} \leq \overline{E}_2 \quad \text{and} \quad \lim_{\overline{E} \longrightarrow \overline{E}_2} g(\overline{E}) = +\infty \tag{A31}$$

Hence,

$$\lim_{\overline{E} \longrightarrow \overline{E}_2^-} G(\overline{E}) = +\infty \quad \text{and} \quad \lim_{\overline{E} \longrightarrow \overline{E}_1^+} G(\overline{E}) = -\infty \tag{A32}$$

Moreover, it is easy to check that $G(\overline{E})$ is a monotonically increasing function of $\overline{E}$ for $\overline{E}_1 \leq \overline{E} \leq \overline{E}_2$. This, together with Result (A32), implies that Equation (A29) has unique solution $\overline{E}(h)$, satisfying $\overline{E}_1 \leq \overline{E} \leq \overline{E}_2$.

- *Now, we will show that $\overline{E}(h)$ is a decreasing function of $h$.*

From Equation (19), using implicit differentiation, we can compute the total derivative of $\overline{E}(h)$ with respect to $h$ as follows:

$$\frac{d\overline{E}}{dh} = -\frac{-\frac{\partial p^e}{\partial h}\overline{E}(1-\alpha) + \frac{\partial p^e}{\partial h}(1-\alpha)(\theta y_L - p^e)}{(1-\alpha)(\theta y_L - p^e) - \frac{p^e}{\partial h}\overline{E}(1-\alpha) + \frac{\partial p^e}{\partial \overline{E}}(1-\alpha)(\theta y_L - p)} \tag{A33}$$

After some arrangements, we obtain:

$$\frac{d\overline{E}}{dh} = \frac{\partial p^e}{\partial h}\frac{NPV - (1-\alpha)\theta y_L + (1-\alpha)(\overline{E} + 2p^e - \rho^*)}{(1-\alpha)(\theta y_L - p^e) - \frac{\partial p^e}{\partial \overline{E}}(NPV - (1-\alpha)\theta y_L + (1-\alpha)(\overline{E} + 2p^e - \rho^*))} \tag{A34}$$

Since $p^e \geq \rho^*$ and $\frac{\partial p^e}{\partial \overline{E}} \leq 0$, from Expression (A34), we see that $\frac{d\overline{E}}{dh}$ has the same sign as $\frac{\partial p^e}{\partial h}$. Since $h$ measures the mass on the left side of the banks' leverage distribution, we have $\frac{\partial p^e}{\partial h}$ as negative. Therefore, $\overline{E}(h)$ is decreasing with $h$.

- *We show that $p\left(\overline{E}(h), h\right)$ is decreasing with respect to both $\overline{E}$ and $h$*

  As shown above, the partial derivative of $p^e$ with respect to $\overline{E}$ is negative, which means that $p\left(\overline{E}(h), h\right)$ is decreasing with $\overline{E}$. Now, we can compute the total derivative of $p^e$ with respect to $h$ as follows:

$$\frac{dp^e}{dh} = \frac{\partial p^e}{\partial \overline{E}} \frac{d\overline{E}}{dh} + \frac{\partial p^e}{\partial h} \tag{A35}$$

  Using Expression (A34), we obtain:

$$\frac{dp^e}{dh} = \frac{\partial p^e}{\partial h} \frac{(1-\alpha)(\theta y_L - p^e)}{(1-\alpha)(\theta y_L - p^e) - \frac{\partial p^e}{\partial \overline{E}} \left[NPV - (1-\alpha)\theta y_L + (1-\alpha)(\overline{E} + 2p^e - \rho^*)\right]} \tag{A36}$$

  Hence, $\frac{dp^e}{dh}$ has the same sign as $\frac{\partial p^e}{\partial h}$, which means that $p\left(\overline{E}(h), h\right)$ is decreasing with $h$.

- *Part (ii) of Proposition 4 is the direct consequence of the fact that $p\left(\overline{E}(h), h\right)$ is decreasing with $h$.*

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